Turbulence Modeling for Three-Dimensional Shear Flows over Curved Rotating Bodies

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It is known that curvature and rotation affect a turbulence structure substantially, and a knowledge of these effects is essential for the improved prediction of flow over rotating bodies. A turbulence model which includes the effects of curvature as well as rotation has been developed. Different hypotheses are introduced to model the higher order unknowns in the Reynolds stress, the turbulent kinetic energy and dissipation rate equations are discussed. A detailed analysis of the effect of the rotation on each component of the Reynolds stress tensor is presented for hypothetical cases such as the pure shear flow in a rotating frame. Calculations show that the effects of rotation on turbulent shear stresses are more pronounced in a centrifugal type of turbomachinery than an axial type.

Nomenclature

C	= constant in turbulence model
D	= velocity gradient ratio, $=(\partial W/\partial y)/(\partial U/\partial y)$
F_{ik}	= viscous stress tensor, $=2\mu\bar{S}_{ik}$
g^{ik}, g_{ik}	= metric tensors
k	= turbulent kinetic energy, $=\frac{1}{2}g^{ik}\overline{u_i u_k}$
ℓ	= turbulent length scale
n_i	= unit vectors
$\bar{p}$	= mean static pressure
p'	= fluctuating static pressure
P	= stress production of k, $=-\overline{u_i u^j}U^i_{,j}$
P_{ik}	= stress production of $\overline{u_i u_k}$, $=-\overline{u_i u^j}U_{k,j}-\overline{u_k u^j}U_{i,j}$
R, R_{ic}	= Richardson number of rotation, Eqs. (9) and (28)
R_{ik}	= Coriolis production of $\overline{u_i u_k}$, Eq. (26)
$\bar{S}_{ik}$	= mean stress tensor, $=\frac{1}{2}(U_{i,k}+U_{k,i})$
S_{ik}	= fluctuating strain tensor, $=\frac{1}{2}(u_{i,k}+u_{k,i})$
T_{ik}	= Reynolds stress tensor, $=\overline{u_i u_k}/k$
R_e, R_T	= turbulent Reynolds numbers
U, V, W	= mean velocity components in x, y, and z directions, respectively
U_i, U^j	= mean velocity components in generalized tensor
u_i, u^i	= fluctuating velocity components in generalized tensor
x_i, x^i	= covariant and contravariant coordinates
x, y, z	= Cartesian coordinates
δ_i^k	= Kronecker tensor
δ	= boundary-layer thickness
ϵ	= dissipation rate of k, $=2\mu\overline{S_{ij}S^{ij}}$
ϵ_{ipj}	= permutation tensor
μ	= molecular viscosity
μ_T	= eddy viscosity
ν	= kinematic molecular viscosity
ρ	= density
ϕ	= stagger angle (Fig. 1a)
ϕ_{ik}	= pressure strain correlation, $=\overline{p' u_{i,k}}$
ψ	= angle of rotation vector in x,y plane
Ω_i, Ω^i	= components of rotation in generalized tensor

Subscript

i	= covariant derivative

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Introduction

PRESENTLY, there is considerable interest in developing numerical procedures to solve the equations governing three-dimensional fluid flows. In particular, improved calculations of viscous flows on rotating bodies, such as turbomachinery rotors, are of great importance. The analysis of such flows is complex, due to the fact that additional effects such as Coriolis and centrifugal forces change the structure of the turbulence, thus invalidating most of the turbulence models that are presently used in computing nonrotating turbulent flows. Moreover, as can be seen in the results of Bradshaw[1] and Johnston et al.,[2] the curvature and/or rotation may affect the stability of the boundary layer and an increase or suppression of the turbulence may occur. It appears, therefore, that assumptions based on the well-known isotropic eddy-viscosity formulation may not be valid since the Reynolds stress tensor is not aligned with the mean strain tensor when additional production or destruction of turbulence is coupled with the production due to mean shear.

Nearly all approaches to account for the streamline curvature effects in the prediction schemes have involved Bradshaw's[1] modification for the mixing length. This treatment has proved to be reasonably successful but is limited by the need to prescribe different empirical constants for different flows. More recently, Gibson and Rodi[3] have proposed a full Reynolds stress model for two-dimensional curved flows. This model takes into account the effect of curvature on stresses.

Very few attempts have been made to introduce the effects of rotation on turbulence. Lakshminarayana and Reynolds,[4] in a qualitative analysis of the effects of rotation on turbulence in the near wake of a rotor, indicated that the rotation may have substantial effects on the structure of the turbulence. Cousteix and Aupoix[5] discussed the effect of the rotation number on the behavior of turbulent stresses for a specific case. Bertoglio[6] carried out a spectral analysis to study the effects of the rotation on a homogeneous turbulent field. A complete calculation of the three-dimensional rotating flow in a square duct has been performed by Howard et al.[7] using a turbulence model based on the two-equation k-ϵ model of Ref. 8. The authors emphasize the need for inclusion of the effects of the Coriolis forces on the turbulence structure. Hah and Lakshminarayana[9] solved the governing equations of a rotor wake using a k-ϵ model and an approximation of the Reynolds stress equations. They showed good agreement with the mean velocity and turbulence quantities. So[10] has carried out modeling of curved shear flows for two-dimensional shear flows (with application to geophysical flow) by assuming that the production balances the viscous dissipation in the Reynolds stress equation.

The work presented in this paper is part of a program motivated by the requirement for a general prediction procedure for complex shear flows with rotation and streamline curvature. The major objective of this paper is the analysis of the effect of rotation on the dissipation rate equation and the Reynolds stress equations.

Governing Equations and Modeling

The modifications of the turbulent structure due to the rotation can be analyzed clearly by utilizing the equations in a relative frame of reference. Most of the curvature effects will be implicitly included in the equations written in a generalized tensor form. The equations of mean and turbulent quantities for an incompressible flow are presented in conservative form below.

The continuity equation is given by

$$U^i{}_{,i}=0; \qquad u^i{}_{,i}=0 \tag{1}$$

where U^i and u^i represent, respectively, the contravariant components of mean and fluctuating velocity.

The momentum equation is given by

$$\begin{aligned}(\rho U_i U^j)_{,j} &= -2\epsilon_{ipj}\rho\Omega^p U^j - \rho[(\Omega_j x^j)\Omega_i - (\Omega_j\Omega^j)x_i] \\ &\quad - (\bar{p}\delta^j_i + \rho\overline{u_i u^j} - \overline{F^j_i})_{,j}\end{aligned} \tag{2}$$

where the first and second terms on the right-hand side are the Coriolis and centrifugal forces, respectively. ϵ_{ipj} is the permutation tensor, Ω^P the components of the rotation vector, x^i and x_i the contravariant and covariant coordinates.

The Reynolds stress equation is given by

$$\begin{aligned}(\rho\overline{u_i u_k}U^j)_{,j} &= -(\overline{p'u_i}\delta^j_k + \overline{p'u_k}\delta^j_i + \rho\overline{u_i u_k u^j} \\ &\quad - \overline{u_i F^j_k} - \overline{u_k F^j_i})_{,j} + (\overline{p'u_{i,k}} + \overline{p'u_{k,i}}) \\ &\quad - (\rho\overline{u_i u^j}U_{k,j} + \rho\overline{u_k u^j}U_{i,j}) - (\overline{u_{i,j}F^j_k} + \overline{u_{k,j}F^j_i}) \\ &\quad - 2\rho\Omega^P(\epsilon_{ipj}\overline{u_k u^j} + \epsilon_{kpj}\overline{u_i u^j})\end{aligned} \tag{3}$$

where p' is the fluctuating pressure.

The kinetic energy equation is given by

$$(\rho k U^j)_{,j} = -(\overline{u_i p'}\delta^{ij} + \rho\overline{k u^j} - \overline{u_i F^{ij}})_{,j} - \rho\overline{u^i u^j}U_{i,j} - \overline{F^{ij}u_{i,j}} \tag{4}$$

where $\overline{F^{ij}u_{i,j}} = \rho\epsilon = 2\mu\overline{S_{ij}S^{ij}}$ is the dissipation rate, $k = \frac{1}{2}g^{ik}\overline{u_i u_k}$ is the turbulent kinetic energy, and g^{ik} is the metric tensor.

The equation for the dissipation rate is given by

$$\begin{aligned}(\rho\epsilon U^j)_{,j} &= -4\mu\overline{S_{ik}u^j}\bar{S}^{ik}_{,j} - (\rho\overline{\epsilon u^j})_{,j} + g^{nj}\mu\epsilon_{,nj} \\ &\quad - 4\mu U^j{}_{,k}\overline{S^{ik}u_{i,j}} - 4\mu U_{i,j}\overline{S^{ik}u^j{}_{,k}} - 4\mu\overline{S^{ik}u_{i,j}u^j{}_{,k}} \\ &\quad - 4\mu\nu g^{nj}\overline{S^{ik}_{,j}S_{ik,n}} - 4\nu\overline{S^{ik}p'_{,ik}} - 8\mu\epsilon_{ipj}\Omega^P\overline{S^{ik}u^j{}_{,k}}\end{aligned} \tag{5}$$

We may note that Eq. (5) is not independent in regard to the transformation from a noninertial frame of reference to a rotating frame of reference.

In the momentum equation, the rotation effects appear through the Coriolis and centrifugal forces, but it also modifies the mean flowfield through the Reynolds stresses.

k and ϵ Equations

It is interesting to note that the rotation term vanishes identically in the equation of kinetic energy k. However, the rotation will affect the kinetic energy principally through the production by the mean strain, and also through the dissipation. The production term in Eq. (4) depends on how the Reynolds stresses are modeled and the procedure used to obtain the dissipation term from Eq. (5). Different authors have pointed out the difficulty in solving Eq. (5). An order of magnitude analysis of the terms in Eq. (5) is carried out in Ref. 11. The results indicate that the effect of rotation through the explicit term is on the order $(R_e)^{-1/2}$ ($R_e = u'\ell/\nu$, where u' and ℓ are, respectively, fluctuating velocity and a length scale of the gross structure), and for high Reynolds number of turbulence, it is negligible compared with other source/sink terms. In fact, in the latter case, the dissipation processes occur almost only at high wave numbers. Therefore, an isotropy hypothesis for the dissipation may represent the small structures quite well.[12] For low turbulent Reynolds number flows, the viscous diffusion may not be neglected. Moreover, very near a wall, the dissipation processes are not isotropic. In the present work we adopt the following form of Eqs. (4) and (5) developed in Ref. 11, which is similar to Jones and Launder's k-ϵ model for two-dimensional shear flows.

$$\begin{aligned}(\rho k U^j)_{,j} &= \left[\left(\mu + \frac{\mu_T}{c_k}\right)g^{ij}k_{,i}\right]_{,j} - \rho\overline{u^i u^j}U_{i,j} \\ &\quad - \rho\epsilon - \frac{10}{3}g^{ij}\mu k^{1/2}_{,i}\, k^{1/2}_{,j}\end{aligned} \tag{6}$$

The dissipation rate is given by

$$\begin{aligned}(\rho\epsilon U^j)_{,j} &= \left[\left(\mu + \frac{\mu_T}{C_\epsilon}\right)g^{ij}\epsilon_{,i}\right]_{,j} - C_{\epsilon_2}\rho\frac{\epsilon^2}{k} \\ &\quad - C_{\epsilon_1}F_\epsilon(R_T,R_{ic})\rho\frac{\epsilon}{k}\overline{u^i u^j}U_{i,j} + C_{\epsilon_3}\mu\frac{k^2}{\epsilon}g^{ij}(\bar{S}_{\ell k,i}\bar{S}^{\ell K}_{,j})\end{aligned} \tag{7}$$

where $C_k, C_\epsilon, C_{\epsilon 1}, C_{\epsilon 2}, C_{\epsilon 3}$ are modeling constants and F_ϵ is given by the relation

$$F_\epsilon(R_T,R_{ic}) = 1 + 0.3(1-R_{ic})\exp(-R_T^2) \tag{8}$$

where

$$R_T = \frac{k^2}{\nu\epsilon}, \qquad R_{ic} = -2\frac{\epsilon_{ipj}\Omega^P}{U_{i,j}} \tag{9}$$

The eddy viscosity is $\mu_T = C_\mu f_\mu\ (\rho k^2/\epsilon)$ and $C_\mu = \exp\{-3.4/[1+(R_T/50)^2]\}$. The last term on the right-hand side of Eq. (6) represents the value of ϵ at the wall which is generally not zero.[8,11] The last term in the right-hand side of Eq. (7) is negligible at high turbulent Reynolds number. It represents the production by mean strain derivatives. It is remarkable that this term can be considered as a generalization of a similar term proposed in Ref. 8 for the calculation of two-dimensional boundary layers including the viscous sublayer. It is also important to note that the last terms in Eqs. (6) and (7) are important only in the vicinity of the wall. As far as the rotation is concerned, it appears that it will mainly affect the production terms in Eq. (6). In fact, if we examine Eq. (3) it is easy to show that the Reynolds stresses may be greatly affected by the rotation, while the kinetic energy and its dissipation rate are not influenced much by the rotation. The analysis of the dissipation equation shows that the rotation should affect the "production" term instead of the "dissipation" term. Most of the models up to now account for the rotation through the dissipation term.[7,13] Nevertheless, the present analysis seems to be in accord with

Launder et al.[13] who argue that the corrections might be made better on the production term of the dissipation equation rather than the decay part.

As the explicit effect of the Richardson number of rotation appears only at low Reynolds number (e.g., near the walls), it is likely that the rotation will not affect the dissipation rate substantially, except near the walls.

Algebraic Reynolds Stress Equations

It is necessary to develop a suitable model for the Reynolds stresses. For simple shear flows an isotropic eddy viscosity relating the Reynolds stress tensor to the mean shear strain tensor gives quite reasonable results. However, for three-dimensional shear flows, the velocity vectors $\hat{U}$ and $\nabla \hat{U}$ are not always aligned; therefore, the eddy viscosity leads to unrealistic simulations.

Models employing transport equations for the individual turbulent stresses constitute a large number of nonlinear differential equations. Hence, these equations have to be modeled to provide for a manageable set of equations. The transport and diffusion terms in Eq. (3) are treated by the technique known as algebraic stress modeling.[11,14] The net transport of $\overline{u_i u_k}$ is assumed to be locally proportional to the net transport of k, the proportionality factor being $\overline{u_i u_k}/k$. Therefore, this assumption leads to the following equation:

$$(\rho \overline{u_i u_k} U^j)_{,j} - \mathrm{dif}(\overline{u_i u_k}) = \frac{\overline{u_i u_k}}{k} \left[(\rho k U^j)_{,j} - \mathrm{dif}(k) \right] \tag{10}$$

where $\mathrm{dif}(\overline{u_i u_k})$ and $\mathrm{dif}(k)$ denote the respective diffusion terms. Equation (10) then may be written as follows:

$$S(\overline{u_i u_k}) = \frac{\overline{u_i u_k}}{k} S(k) \tag{11}$$

where $S(\overline{u_i u_k})$ and $S(k)$ represent, respectively, the source/sink terms of the Reynolds stresses components and turbulent kinetic energy equations [Eqs. (3) and (4)].

$$S(\overline{u_i u_k}) = -(\rho \overline{u_i u^j} U_{k,j} + \rho \overline{u_k u^j} U_{i,j}) - 2\rho \Omega^P (\epsilon_{ipj} \overline{u_k u^j} + \epsilon_{kpj} \overline{u_i u^j}) + (\overline{p' u_{i,k}} + \overline{p' u_{k,i}}) - (\overline{F^j_k u_{i,j}} + \overline{F^j_i u_{k,j}}) \tag{12}$$

$$S(k) = -\rho \overline{u^i u^j} U_{i,j} - \rho\epsilon - \frac{10}{3} g^{ij} \mu k^{1/2}_{,i} k^{1/2}_{,j} \tag{13}$$

The two last terms in Eq. (12) remain to be modeled. The first term is the so-called pressure-strain correlation, and the second term represents the dissipative effects.

The modeling of pressure strain correlation has been widely discussed[15,16] for nonrotating flows. It can be shown that two kinds of interaction give rise to these correlations, one involving only turbulence quantities and another arising from the presence of the mean strain rate and the body forces.

$$\overline{p' u_{i,k}} = \phi_{ik,1} + \phi_{ik,2} \tag{14}$$

A detailed analysis of the two components of $\overline{p' u_{i,k}}$, when curvatures and rotation are present, is given in Ref. 11. A summary is given below. The pressure strain correlation at position x in the flow may be obtained through the integration of a Poisson equation for the pressure fluctuation correlated with the fluctuating strain around the position x. The first component is a symmetric tensor with zero trace which vanishes if the turbulence is isotropic. Therefore, it is natural to express it in terms of the anisotropy tensor of the turbulence.[17]

$$\phi_{ik,1} + \phi_{ki,1} = -C_1 \rho \frac{\epsilon}{k} \left(\overline{u_i u_k} - \frac{2}{3} \delta_{ik} k \right) \tag{15}$$

Following the procedure used for the stationary flows (e.g., Ref. 17), it can be shown that the "rapid term" is given by[22]

$$\phi_{ik,2} + \phi_{ki,2} - C_2 \left(P^*_{ik} - \frac{2}{3} P \delta_{ik} \right) \tag{16}$$

where

$$P^*_{ik} = -\rho \overline{u_i u^j} U^*_{k,j} - \rho \overline{u_k u^j} U^*_{i,j}, \qquad P = -\rho \overline{u^\ell u^m} U_{\ell,m}$$

$$U^*_{k,j} = U_{k,j} + \epsilon_{kpj} \Omega^P \tag{17}$$

It can be shown that the vicinity of the wall may affect the pressure strain correlation significantly as long as the typical size of the energy containing eddies is on the order of the distance from the wall. This condition is always satisfied in near-wall flows. In Ref. 16 it is shown how the pressure strain correlation may be modified because of the contribution of the near-wall effect corresponding to the reflecting wall influence of $\phi_{ik,1}$ and $\phi_{ik,2}$. The pressure strain correlation is then modeled as

$$\overline{p' u_{i,k}} + \overline{p' u_{k,i}} = \left\{ -C_1 \rho \frac{\epsilon}{k} \left(\overline{u_i u_k} - \frac{2}{3} \delta_{ik} k \right) - C_2 \left(P^*_{ik} - \frac{2}{3} \delta_{ik} P \right) \right\} F\left(\frac{\ell}{n_i x^i} \right) \tag{18}$$

where C_1 and C_2 are modeling constants.

The wall function $F(\ell / n_i x^i)$ is given by

$$F\left(\frac{\ell}{n_i x^i} \right) = 1 + 2.5 \frac{k^{3/2}}{\epsilon n_i x^i} \tag{19}$$

The remaining unknown correlations in Eq. (12) are the dissipative terms

$$D = -(\overline{F^j_k u_{i,j}} + \overline{F^j_i u_{k,j}}) \tag{20}$$

In incompressible flows, where only the fluctuating quantities remain, D becomes

$$D = -\mu (\overline{S^j_k u_{i,j}} + \overline{S^j_i u_{k,j}}) \tag{21}$$

Terms such as $\overline{S^j_k u_{i,j}}$ and $\overline{S^j_i u_{k,j}}$ are functions of a number of parameters such as ϵ, $\bar{S}_{ik}$, and ν. Therefore, these terms may be modeled as follows[11,18]:

$$\overline{S^j_k u_{i,j}} \approx \frac{\epsilon}{3\nu} \left\{ \delta_{ik} + \alpha \sqrt{\frac{\nu}{\epsilon}} \bar{S}_{ik} \right\}$$

$$\overline{S^j_i u_{k,j}} \approx \frac{\epsilon}{3\nu} \left\{ \delta_{ik} + \alpha \sqrt{\frac{\nu}{\epsilon}} \bar{S}_{ik} \right\} \tag{22}$$

where α is a constant.

An order of magnitude analysis shows that the term involving the mean strain rate is of the order $R_T^{-1/2}$, and, therefore, is negligible in high turbulent Reynolds number flows. In this case the decay rate D reduces to

$$D = -\frac{2}{3} \delta_{ik} \rho \epsilon \tag{23}$$

which is the form proposed in Ref. 16, where it is assumed that the dissipative motions are locally isotropic. Several experimental studies have shown that turbulence does not remain locally isotropic in the presence of strong strain fields. In the case of low Reynolds number flows, the energy-containing and dissipation range of frequencies overlap and the anisotropy introduced by the strain field has to be included. Therefore, the dissipation term in Eq. (12) is modeled as follows:

$$D = -\frac{2}{3}\rho\epsilon^*\left(\delta_{ik} + \frac{k}{\epsilon}R_T^{-1/2}\bar{S}_{ik}\right) \tag{24}$$

Substitution of the model assumption equations (18) and (24) into Eq. (11) yields the following algebraic equation for the Reynolds stresses:

$$\frac{\overline{u_i u_k}}{k} = \frac{2}{3}\delta_{ik} + \frac{R_{ik}\left(1-\frac{C_2}{2}F\right) + \left(P_{ik} - \frac{2}{3}\delta_{ik}P\right)(1-C_2F)}{P+\rho\epsilon^*(C_1F-1)} - \frac{\frac{2}{3}\rho k R_T^{-1/2}\bar{S}_{ik}}{P+\rho\epsilon^*(C_1F-1)} \tag{25}$$

where

$$\epsilon^* = \epsilon + \frac{10}{3}g^{lm}\nu k^{1/2}_{,l}\, k^{1/2}_{,m}$$

$$P_{ik} = -\rho\left(\overline{u_i u^j}\,U_{k,j} + \overline{u_k u^j}\,U_{i,j}\right)$$

$$R_{ik} = -2\rho\Omega^P\left(\epsilon_{ipj}\overline{u_k u^j} + \epsilon_{kpj}\overline{u_i u^j}\right) \tag{26}$$

and F is the function defined in Eq. (19). This algebraic Reynolds stress model is suitable whenever the transport of $\overline{u_i u_k}$ is not very important. On the other hand, all effects that enter the transport equations for $\overline{u_i u_k}$ are accounted for by this model as, for example, the rotation, the streamline curvature, nonisotropic strain fields, and the wall damping influence. Therefore, the model can simulate many of the flows associated with engineering application, such as boundary layers, mixing layers, etc.

Procedure for Implementing the Turbulence Closure Scheme

The combination of Eqs. (6), (7), and (25), together with the momentum and continuity Eqs. (2) and (1) form a closed set, as long as the seven modeling constants are determined. The empirical coefficients are as follows: $C_k = 1$; $C_\epsilon = 1.3$; $C_{\epsilon_1} = 1.44$; $C_{\epsilon_2} = 1.92$; $C_{\epsilon_3} = 0.2$; $C_1 = 1.5$; and $C_2 = 0.6$.

To solve the governing equations of the mean and turbulent quantities simultaneously for three-dimensional flows over complicated bodies involves a great amount of computation. Time- and space-marching methods seem promising for such computations. However, these schemes are still under development. It is useful to know the response of the turbulence model to certain flow configurations. Considering the mean field as a known parameter, it is then possible to compute the turbulent stresses. An attempt has been made in the next section to isolate the effects of rotation on turbulence properties utilizing the algebraic stress equations [Eq. (25)].

Results and Discussion

The Coriolis forces associated with the rotation contribute to the production of Reynolds stress through the term R_{ik} in Eq. (25). Depending on the direction and magnitude of the rotation vector, the Reynolds stress tensor components will be affected differently. The effects of curvature and rotation act simultaneously on the Reynolds stress tensor resulting in either augmentation or suppression of turbulence intensities and stresses. In the present study, the response of the set of six nonlinear algebraic Reynolds stress equation (25) to a change in the rotation number (intensity and direction) is analyzed. The flow is assumed to be in local equilibrium (e.g., the production of turbulent kinetic energy is locally equal to the dissipation rate, $P = \epsilon$). Such situations are representative of the flow encountered in thin shear layer flows, for example, blade boundary layers in turbomachinery. The effects of the curvature and the near wall are not included in this analysis in order to isolate the effects due to the rotation only. Hence, results should be interpreted with caution as they may not truly simulate the flow in a practical turbomachinery. However, the present calculations provide valuable insight concerning the evolution of turbulence under the influence of rotation.

With the assumptions introduced above, Eq. (25) reduces, in Cartesian coordinates, to

$$T_{ik} = \frac{2}{3}\delta_{ik} + \left\{\left[R_{ik}\left(1-\frac{C_2}{2}\right) + \left(P_{ik} - \frac{2}{3}\delta_{ik}P\right)(1-C_2)\right]\Big/C_1P\right\} \tag{27}$$

which represents a set of six equations with functions of the mean velocity gradients, rotation vector, and Reynolds stresses. The nature of the flows that exist on axial and centrifugal turbomachinery blades are shown in Figs. 1a and 1b, respectively. These configurations are investigated. In a three-dimensional thin shear layer, the boundary-layer approximations are generally valid. Therefore, only the normal velocity gradients $\partial U/\partial y$ and $\partial W/\partial y$ are retained in Eq. (27) together with the rotation vector. The following parameters are used in the analysis:

$$D = \frac{\partial W}{\partial y}\Big/\frac{\partial U}{\partial y} \quad \text{and} \quad R = -2\Omega\Big/\frac{\partial U}{\partial y} \tag{28}$$

The first parameter is the ratio of the derivatives of the cross-flow velocity profile to the streamwise velocity profile and indicates the local skewing of the mean strain. The second parameter is the Richardson number of rotation and represents the effect of the Coriolis force on the shear flow. Johnston et al.[2] have shown how this parameter can be related to the flow stability in a rotating duct. Equation (27) leads to a set of six nonlinear equations in T_{ik} expressed in terms of the rotation parameter R and the skewing parameter D. T_{ik} are, respectively, the turbulence intensities ($i=k$) and the shear stresses ($i \neq k$), normalized by the turbulent kinetic energy k. In the case of a flow over a blade in an axial turbomachinery, the angle ϕ between the notation vector and the x axis (Fig. 1a) acts as a third parameter. The Newton-Raphson technique was used to solve the system of six nonlinear equations. The parameters D and R and the angle ϕ were varied during the calculations. The equations can be found in Ref. 22.

For a nonrotating configuration the solution of such a system depends only on the parameter D. When $D = 0$ (i.e.,

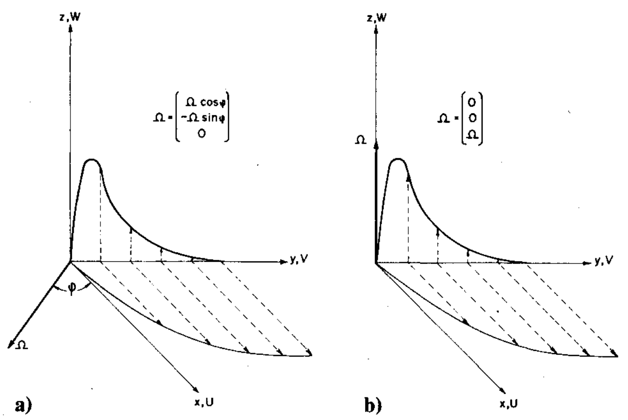


Fig. 1 Nature of velocity profiles: a) axial flow turbomachinery blade, b) centrifugal type of turbomachinery.

Table 1 Comparison of measured and calculated Reynolds stresses in plane equilibrium shear flows, $P/\epsilon = 1$

	T_{xx}	T_{yy}	T_{zz}	$-T_{xy}$
Experimental data (Ref. 19)	0.97	0.49	0.54	0.33
Model results (Ref. 20, $C_l = 1.8$)	0.96	0.52	0.52	0.34
Model results (Ref. 5, $C_l = 1.5$)	0.925	0.465	0.64	0.36
Present results ($C_l = 1.5$)	1.02	0.49	0.49	0.36

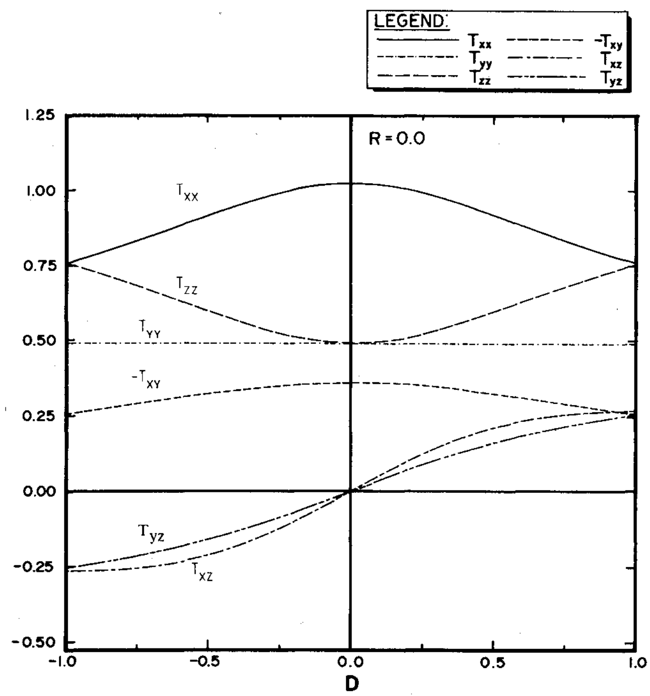


Fig. 2 Variation of Reynolds stress tensor with D.

$\partial W/\partial y = 0$), the stress levels are obtained for local equilibrium turbulence ($P = \epsilon$) in a two-dimensional plane shear flow. The results are shown in Table 1 and are compared with the results of Refs. 5, 19, and 20 for the two-dimensional case. Calculations with present and earlier models agree well with the measurements of Champagne et al.[19]

It is interesting to note that none of the models can predict the differences observed between the normal stresses T_{yy} and T_{zz}. This is inherent in the way the "rapid term" of the pressure-strain correlation [Eq. (18)] is modeled. Cousteix et al.,[5] using a different model for the rapid term,[16] have predicted these differences. However, their results show discrepancies of up to 15% with the experimental data.

The three-dimensional boundary layers occur on many nonrotating bodies (i.e., aircraft wings, stator blades, etc.). It is interesting to understand the response of the turbulence model for varying cross-flow velocity profiles (parameter D). The change of the Reynolds stress tensor with the parameter D is shown in Fig. 2 for the case of $R = 0$ (no rotation). The turbulent kinetic energy is redistributed between the streamwise and crosswise intensities. As the cross-flow gradient increases the energy passes from the streamwise to the crosswise component, while the normal intensity remains unaffected. Similarly, the streamwise shear stress decreases (up to 20% of the two-dimensional value), while the crosswise as well as the normal shear stresses increase.

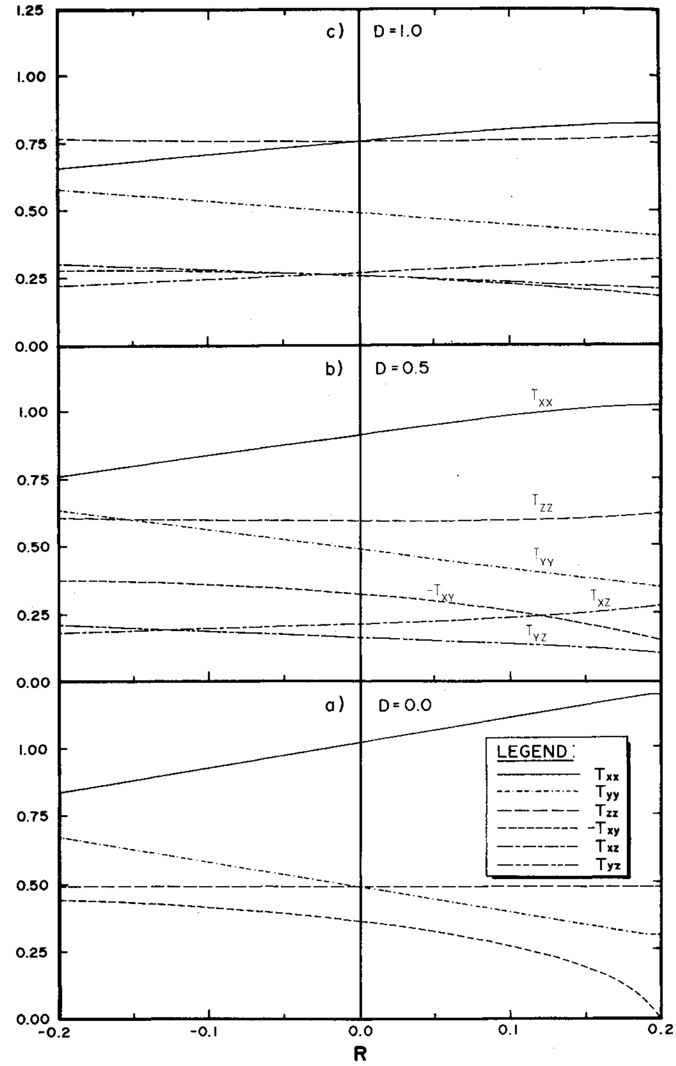


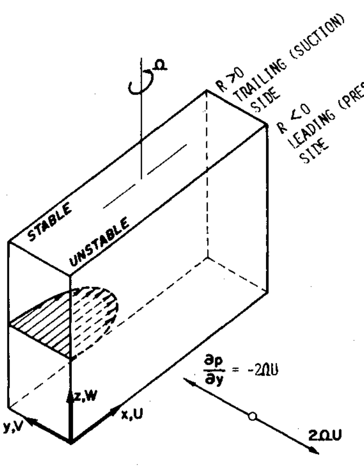
Fig. 3 Effect of rotation on Reynolds stress tensor for centrifugal type of turbomachinery.

Results for the Model Representative of Centrifugal Turbomachinery

The effect of rotation on the Reynolds stress tensor, calculated for the configuration shown in Fig. 1b, is shown in Fig. 3. It is known[2] that the normal intensity T_{yy} and the shear stress $-T_{xy}$ are amplified due to the rotation on the leading side[‡] of the channel, a model representative of the centrifugal type of turbomachinery. The Richardson number of rotation on this side is negative. Hence, it is clear (Fig. 3a) that the phenomena observed by Johnston et al.[2] are predicted qualitatively by the present model. The amplification of the normal intensity T_{yy} is balanced by a decrease in the streamwise intensity T_{xx}. When R is positive (corresponding to the trailing side[‡] of the channel, Fig. 4), the direct effect of rotation is to diminish the shear stress $-T_{xy}$ and redistribute the energy from the normal intensity T_{yy} to the streamwise intensity T_{xx}, compared to the corresponding nonrotating two-dimensional plane flow. In conditions of strong stabilizing effects of the rotation, the shear stress falls to zero at a critical value of R predicted by the model to be 0.2. When a crossflow exists, the redistribution of the energy between each component of intensities follows the same trend as the two-dimensional case ($D = 0$). The rotation still affects the turbulence intensities T_{xx} and T_{yy}. However, the relative levels of changes in intensities are influenced by the cross-flow

‡This terminology is indicated in Fig. 4. The leading (or pressure) side refers to the interior side of the channel which is leading in the direction of rotation.

Fig. 4 Model representative of centrifugal type of turbomachinery.



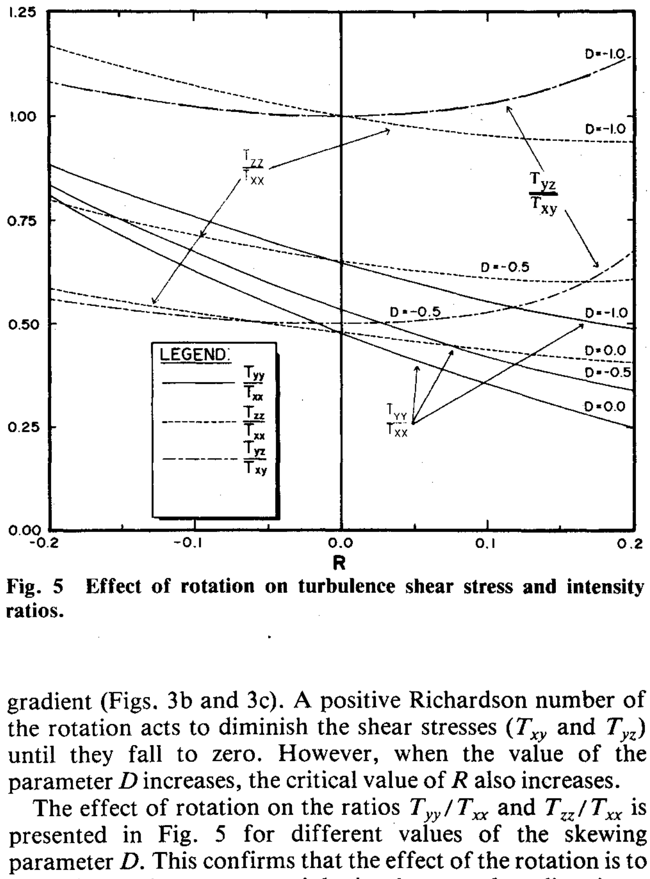


Fig. 5 Effect of rotation on turbulence shear stress and intensity ratios.

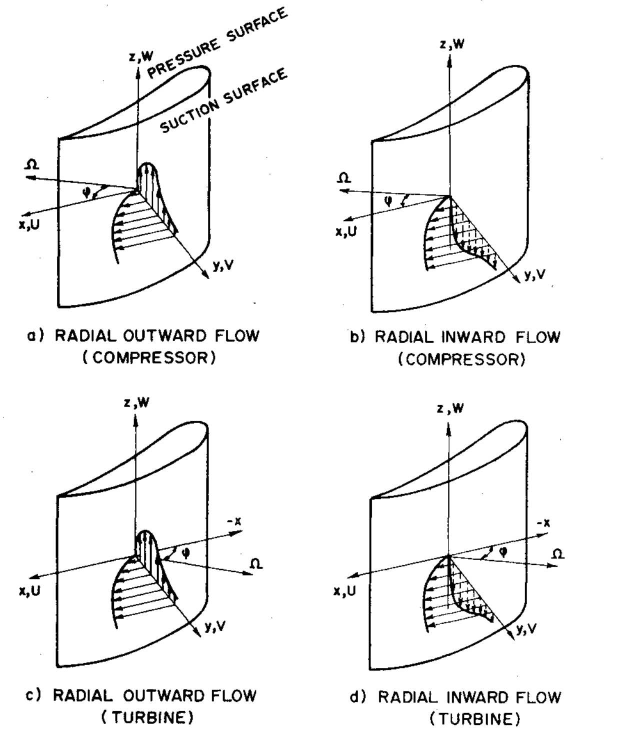


Fig. 6 Schematic of flowfield on rotor blade of axial flow turbomachinery.

gradient (Figs. 3b and 3c). A positive Richardson number of the rotation acts to diminish the shear stresses (T_{xy} and T_{yz}) until they fall to zero. However, when the value of the parameter D increases, the critical value of R also increases.

The effect of rotation on the ratios T_{yy}/T_{xx} and T_{zz}/T_{xx} is presented in Fig. 5 for different values of the skewing parameter D. This confirms that the effect of the rotation is to redistribute the energy mainly in the x and y directions. However, Fig. 5 indicates that the relative effect of the production of turbulence due to the rotation to the production due to the mean shear flow decreases when the parameter D increases (in absolute value). In other words, the rotation affects flows with small cross velocities much more than strong three-dimensional shear flows. In three-dimensional boundary layers, the streamwise and crosswise shear stresses are important as they appear in the U and W momentum equations, respectively. The stress ratio T_{yz}/T_{xy} is also presented in Fig. 5. For the Richardson number of rotation ranging from negative values up to 0.1, the influence of rotation is weak on the ratio of crosswise to streamwise shear stresses. It is remarkable that the ratio T_{yz}/T_{xy} is approximately equal to $-D$. Therefore, it is clear that the stress tensor is nearly aligned with the mean strain tensor and that a simple eddy-viscosity model might be able to capture most features of this flow. However, under conditions of strong stabilizing effects of rotation ($R>0.1$), as the turbulence levels decrease, the relative effect of the rotation to the mean shear strain becomes more important, and the ratio T_{yz}/T_{xy} increases.

Results for Axial Turbomachinery

The four possible configurations for axial flow turbomachinery are illustrated in Fig. 6. The most practical cases are those shown in Fig. 6a (compressor) and Figs. 6c and 6d (turbine). The effect of rotation on the Reynolds stress tensor for $\phi = 45$ deg at various values of D for these cases is shown in Fig. 7. In this figure, plots for $D = -0.1$, 0, and 0.5 represent small crossflow (Figs. 6a and 6c), zero crossflow, and large crossflow, respectively. Since the Coriolis force is in the radial direction, the redistribution of the turbulence intensities occurs between the components T_{xx} and T_{zz}, with very little change in the values of T_{yy}. The effect of rotation is not as pronounced as in the centrifugal type (Fig. 3), where the Coriolis force acts normal to the blade or wall. The influence of rotation on the streamwise stress T_{xy} is small, but its effect on T_{yz} and T_{xz} is substantial.

For zero crossflow ($D=0$) and small crossflow ($D = -0.1$), the redistribution of the turbulence kinetic energy occurs mainly between T_{xx} and T_{zz}, resulting in an increase in the radial (or spanwise) intensities. These cases correspond to the outer layer of blade boundary layers (with small and zero crossflow) illustrated in Figs. 6a and 6c. The negative values of R correspond to the compressor blade (Fig. 6a), and the positive values to the turbine blade (Fig. 6c). The predictions are in qualitative agreement with the rotor blade boundary layer data reported in Ref. 21.

When the crossflow is large ($D=0.5$, Fig. 7), the values of T_{zz} increase and T_{xx} decrease much more rapidly for the negative values of R. The opposite trend exists for the positive values of R. These cases are similar (with the exception of the larger crossflow) to the cases illustrated in Figs. 6d and 6b, respectively. In general, the influence of rotation on turbulence properties is less than those in a centrifugal type of turbomachinery.

The effect of rotation on the ratio of turbulence intensities is shown in Fig. 8 for various values of D and ϕ. It is evident from these figures that the redistribution of energy is mainly

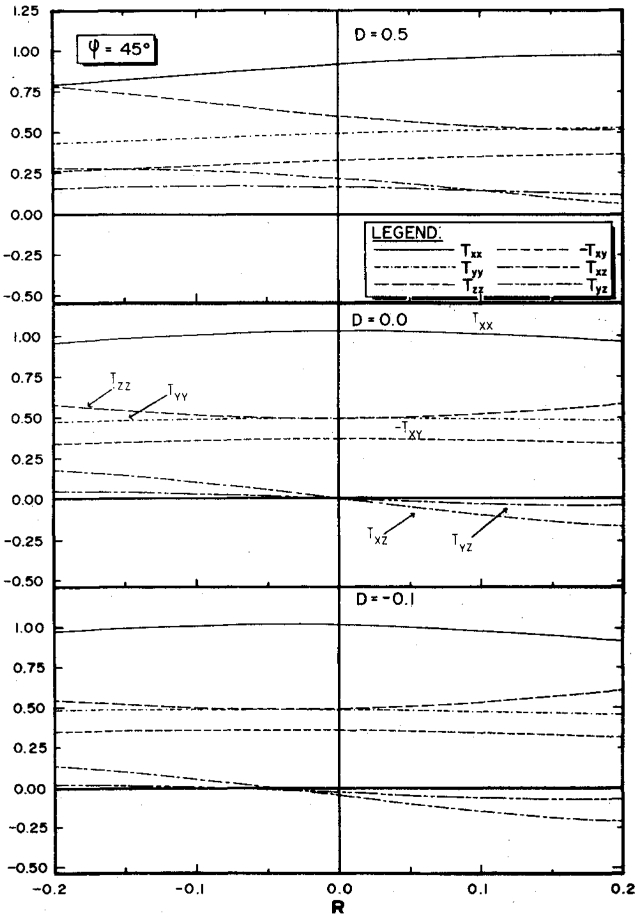


Fig. 7 Effect of rotation on Reynolds stress tensor for axial type of turbomachinery.

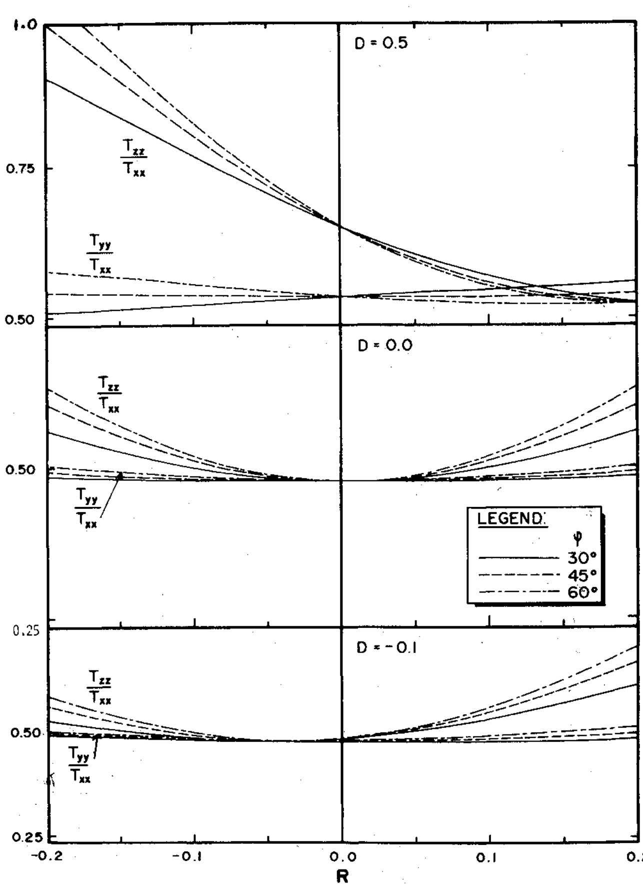


Fig. 8 Effect of rotation on T_{zz}/T_{xx} and T_{yy}/T_{xx} at various values of ϕ and D.

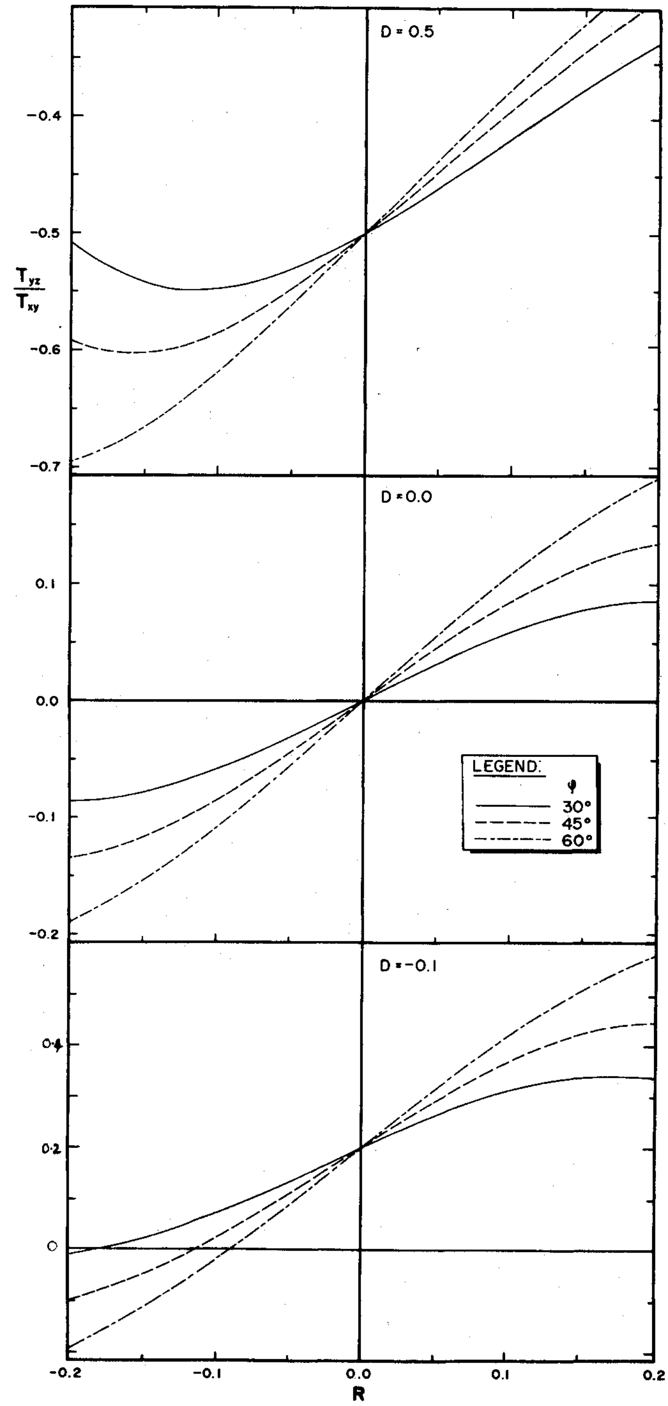


Fig. 9 Effect of rotation on T_{yz}/T_{xy} at various values of ϕ and D.

between T_{xx} and T_{zz}, with very little change in T_{yy}. The rotation has a larger influence at higher stagger angles. For nonzero crossflow, the combined effect of three dimensionality D and the rotation R increases the ratio T_{zz}/T_{xx} for most cases, except for large crossflows and positive values of R. The effect of rotation on the ratio of the spanwise or radial stress to the streamwise stress (T_{yz}/T_{xy}) for various values of ϕ and D are shown in Fig. 9. Figure 7 shows that the streamwise stress T_{xy} is not affected by the rotation, which is very much different from a centrifugal turbomachinery case. Hence, the ratio T_{yz}/T_{xy} is controlled by the values of T_{yz}. The absolute value of the shear stress T_{yz} always increases with an increase in the rotation for the cases shown in Fig. 9, except for $D=-0.1$, R negative and $D=0.5$, positive R. In the latter two cases, the shear stress production due to the crossflow velocity gradient and the rotation are in opposite directions.

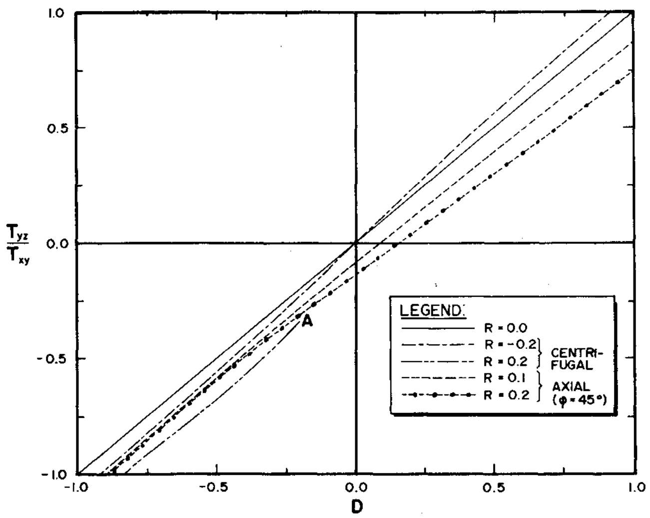


Fig. 10 Variation of T_{yz}/T_{xy} with D at various rotation parameters.

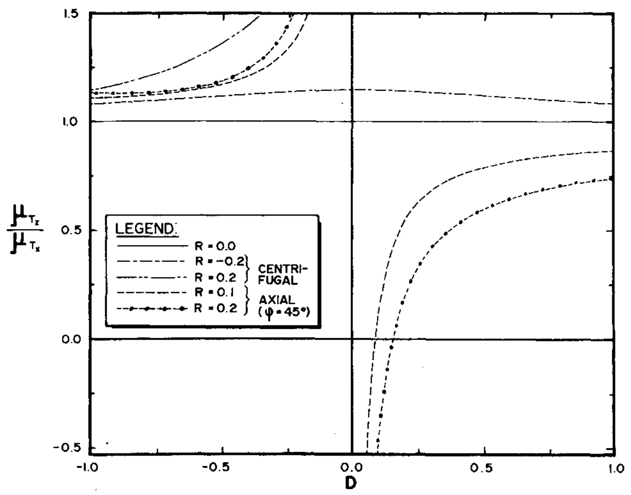


Fig. 11 Ratio of eddy viscosity μ_{Tz}/μ_{Tx}.

The effect of the parameter D on the stress ratio at several Richardson numbers of rotation for both cases (Figs. 1a and 1b) are shown in Fig. 10. When $R=0$, the value of $T_{yz}/T_{xy}=D$, indicating that the eddy viscosity has the same value in either direction as shown in Fig. 11. For the centrifugal type of channel and at $R=0.2$, $T_{xy}=T_{yz}=0$ at point A in Fig. 10 indicates the stabilizing (laminar) effect of the rotation. For $R=-0.2$, the ratio increases with an increase in the value of D. The eddy viscosity in the radial direction (μ_{Tz}) is higher (about 20%) than the value in the axial direction (μ_{Tx}) as shown in Fig. 11. For the axial type of blading, T_{yz} is nonzero, even for $D=0$. Substantial variation in this ratio as well as the ratios of the eddy viscosity (μ_{Tz}/μ_{Tx}) are observed for this case. Thus, it is evident that the eddy-viscosity concept (i.e., its value is scalar being equal in all directions) is not valid when the rotation is present, especially for the axial type of turbomachinery.

Concluding Remarks

The need of a general prediction procedure for complex shear flows with rotation and streamline curvature motivated the work presented in this paper. Since the turbulence model involves two transport equations and six algebraic Reynolds stress equations which contain the essential mechanisms of production and decay of turbulence, the model should be able to predict the main features of the flow. This would necessitate a three-dimensional Navier-Stokes code. The major thrust of this paper has been modeling and analyzing the effect of the rotation on turbulent quantities. The analysis of k and ϵ equations shows that the rotation affects these quantities only indirectly through the production terms (i.e., through the Reynolds stresses). In strong stabilizing rotating flows the turbulence may disappear.

The results from the algebraic Reynolds stress model indicate that in the case of a centrifugal type of turbomachinery the rotation mainly redistributes the energy between the streamwise and the normal (to the blade) intensities. An increase of the Richardson number leads to a decrease of the Reynolds stresses. However, when the crossflow increases this trend is weaker. In other words, strong three-dimensional flows are less affected by the rotation than flows with small cross velocities. The effects of the rotation are not so much pronounced in an axial type of turbomachinery. The redistribution occurs mainly between the streamwise and spanwise intensities. The turbulence stress in the spanwise direction is especially altered. These features are functions of the three dimensionality in the velocity profiles and stagger angle of the blade. The rotation has a larger influence at higher stagger angles. There would be an increase in turbulence energy and stress in the radial direction in most practical cases, even though in several instances the combined influences may have a stabilizing effect. The calculation shows clearly that a scalar representation of eddy viscosity is not valid since the Reynolds stress tensor and mean strain tensor are not aligned.

Future efforts should be directed at testing the model proposed and the conclusion drawn in this paper by a careful and basic experimentation incorporating the rotation and three dimensionality.

Acknowledgments

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